

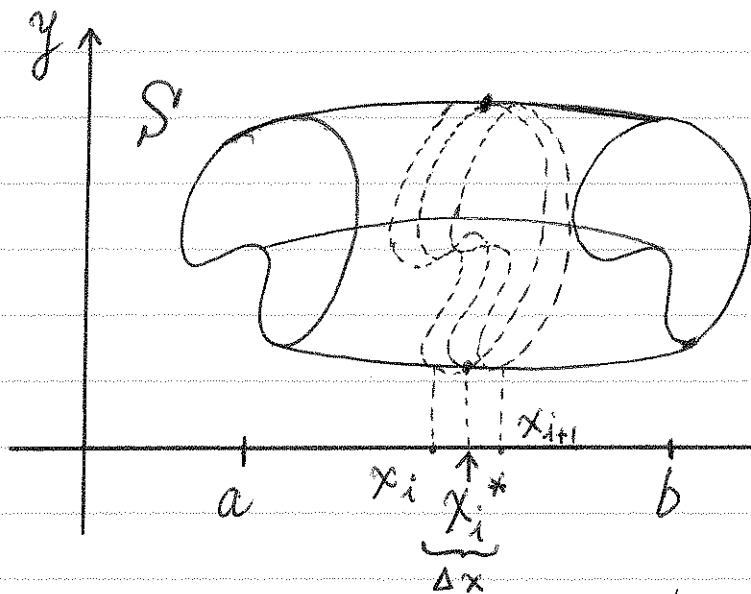
## Section 5.2: Volumes

The goal is to calculate the volume of a 3-dimensional object  $S$ :

- \* cut  $S$  into small thin pieces

- \* approximate every piece by a cylinder

- \* add the volumes of all these cylinders



We divide the interval

$[a, b]$  into  $n$  subintervals

$[x_i, x_{i+1}]$ . On each

subinterval, we take a

vertical cross-section at an intermediate point  $x_i^*$ . We denote

the area of this cross-section  $A(x_i^*)$ . Then the volume of the

thin cylinder at  $x_i^*$  is  $A(x_i^*) \cdot (x_{i+1} - x_i) = A(x_i^*) \cdot \Delta x$ .

For a very small  $\Delta x$ , the sum of the volumes of these thin

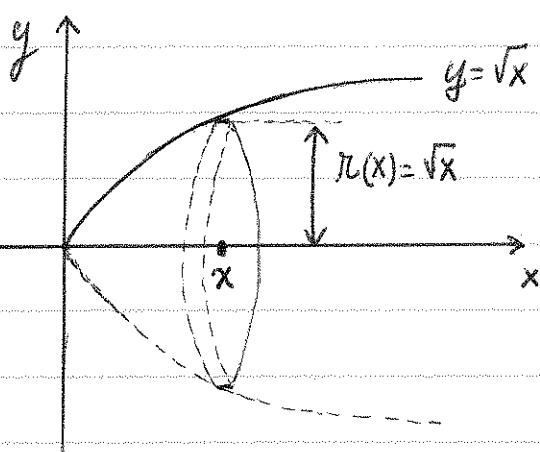
cylinders is a good approximation to the true volume of  $S$

That is, Volume of  $S = \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} A(x_i^*) \cdot \Delta x = \int_a^b A(x) dx$

since the sum above is a Riemann sum.

**Example 1** Find the volume of the solid of revolution, obtained by rotating about the  $x$ -axis, the region under  $y = \sqrt{x}$ , from  $x=0$  to  $x=1$ .

**Solution:**

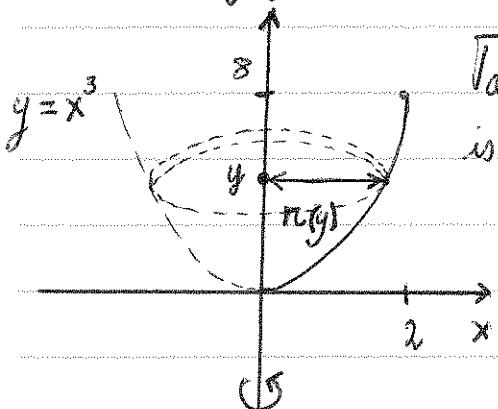


Take a vertical cross-section at some point  $x$  between  $x=0$  and  $x=1$ . The cross-section is circular, with radius  $r(x) = \sqrt{x} - 0 = \sqrt{x}$ . Thus it has area  $A(x) = \pi r^2(x) = \pi x$ .

The volume of the solid is therefore

$$V = \int_0^1 A(x) dx = \int_0^1 \pi x dx = \left[ \frac{\pi}{2} x^2 \right]_0^1 = \frac{\pi}{2}$$

**Example 2** Find the volume of the solid obtained by rotating the region bounded by  $y = x^3$ ,  $y = 8$  and  $x = 0$  about the  $y$ -axis.



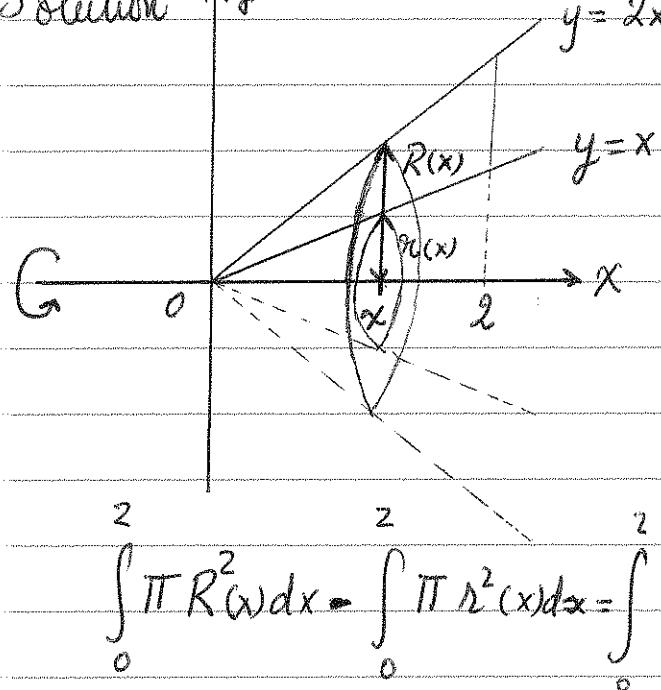
Take a horizontal cross-section at a point  $y$  on the  $y$ -axis. It is circular, with radius  $r(y) = x = \sqrt[3]{y}$ . Therefore

$$\text{Volume} = \int_0^8 A(y) dy = \int_0^8 \pi (\sqrt[3]{y})^2 dy = \frac{96}{5} \pi$$

What if the cross-section is a washer? Here's an example

Example 3 Find the volume of the solid obtained by rotating the region bounded by  $y=x$ ,  $y=2x$ , and  $x=2$  about the  $x$ -axis.

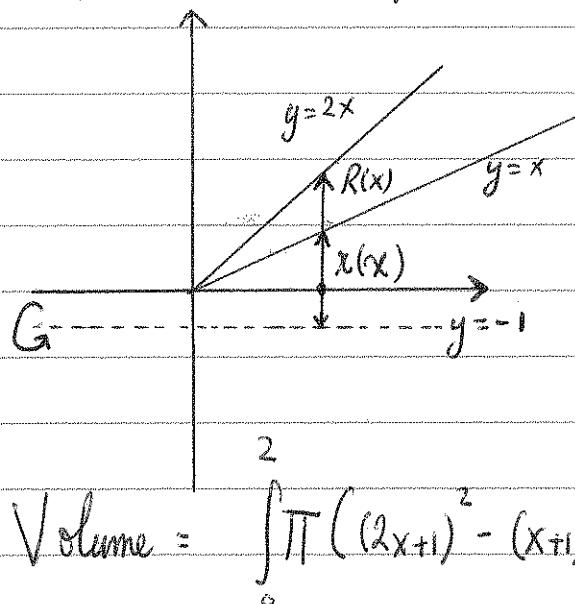
Solution



We think of this volume as Big volume - Small volume. The larger solid has a circular cross-section at  $x$  with outer radius  $R(x) = 2x$ . The smaller solid has a circular cross-section at  $x$  with inner radius  $r(x) = x$ . Thus the volume of the solid is

$$\int_0^2 \pi R^2(x) dx - \int_0^2 \pi r^2(x) dx = \int_0^2 \pi (R^2(x) - r^2(x)) dx = \int_0^2 \pi ((2x)^2 - x^2) dx = 8\pi$$

Example 4: what if the same region as in Ex 3 is rotated about  $y=-1$ ?



We always measure the radius from the axis of rotation. And so

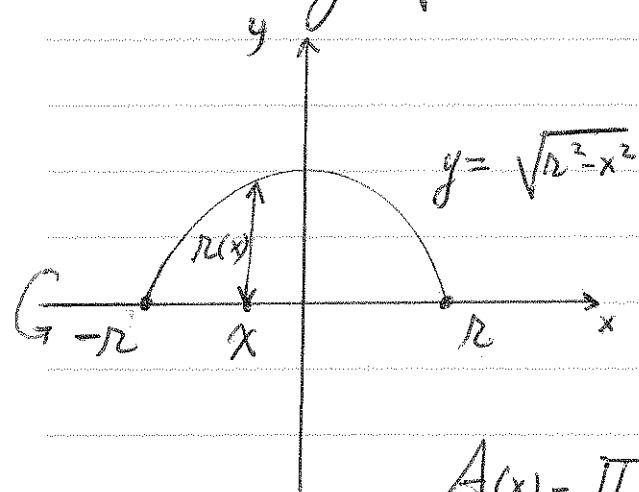
$$r(x) = x - (-1) = x + 1, \text{ and}$$

$$R(x) = 2x - (-1) = 2x + 1; \text{ Therefore}$$

$$\text{Volume} = \int_0^2 \pi ((2x+1)^2 - (x+1)^2) dx = 12\pi$$

Example 5 Derive the formula: Volume of a Sphere of radius  $r$  is  $\frac{4}{3}\pi r^3$ !

Solution: We will think of the sphere of radius  $r$  as the solid generated by rotating the semi-circle  $y = \sqrt{r^2 - x^2}$  of radius  $r$  about the  $x$ -axis.

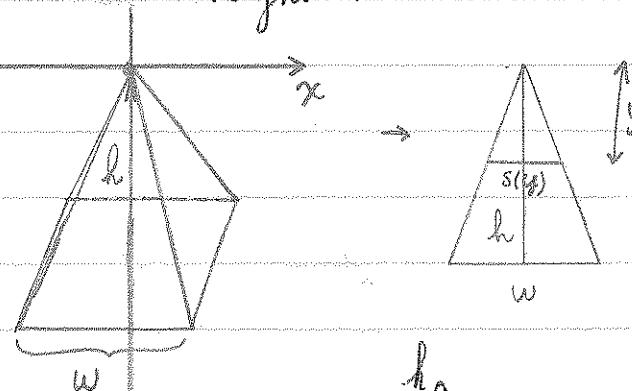


Take a vertical cross-section at a point  $x$  on the  $x$ -axis between  $x = -r$  and  $x = +r$ . The cross-section is circular, with radius  $r(x) = \sqrt{r^2 - x^2}$ , and area

$A(x) = \pi r^2(x) = \pi(r^2 - x^2)$ . The volume of the sphere is therefore

$$\begin{aligned} V &= \int_{-r}^r A(x) dx = \int_{-r}^r \pi(r^2 - x^2) dx = \pi \left( r^2 x - \frac{x^3}{3} \right) \Big|_{-r}^r \\ &= \frac{2}{3} \pi r^3 - \left( -\frac{2}{3} \pi r^3 \right) = \frac{4}{3} \pi r^3. \end{aligned}$$

Example 6. Calculate the volume of a pyramid with a square base of side length  $w$ , and height  $h$ .



Solution: a horizontal cross-section of height  $y$  has a square shape of side length  $s(y)$ .

by similar triangles,  $s(y) = \frac{w}{h} \cdot y$ .

Thus the area of this cross-section is

$$A(y) = s^2(y) = \frac{w^2}{h^2} \cdot y^2. \text{ Therefore, The}$$

$$\text{Volume is } \int_0^h A(y) dy = \int_0^h \frac{w^2}{h^2} y^2 dy = \frac{w^2}{h^2} \cdot \frac{y^3}{3} \Big|_0^h = \frac{1}{3} w^2 h.$$